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BRIEF COMMUNICATION

SHORT NOTE ON THE SPACE AVERAGING IN CONTINUUM MECHANICS

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Abstract—The relationship between averagings over the volume and over the oriented plane cross-section of a heterogeneous medium is considered. It is shown that the claim of the necessary symmetry of the macrostress tensor based on the equivalence of the mentioned methods of averaging is wrong. This problem is crucial for the applications of asymmetric mechanics for description of flows of fluids with suspended rotating particles.

INTRODUCTION

Methods of averaging are often used for the development of field equations of mechanics of continua with complicated microstructure. The fluid with suspended particles is an example of such a continuum. The use of averaging is essential for checking with the alternative approach, i.e. the phenomenological formulation of equations *a priori*, based only on experimental rheological measurements.

This problem is common for continual theories of multiphase fluids, of composite materials, of turbulence with and without suspended particles, etc. A number of papers are devoted to this problem.

During the space averaging of field equations which are valid in microscale, two types of averaged values appear, those averaged over the volume and those averaged over the surfaces. Are they equal to each other in general or not? The answer is essential for utility of the Cosserat mechanics.

IS THE ASYMMETRY OF STRESS TENSOR IMAGINARY?

Nigmatulin (1978) has studied the problem of the space averaging in three chapters (more than 130 pages) of his book on mechanics of heterogeneous media. The central role of these chapters is the statement of the general equivalence (p. 65) of the mean values averaged over a volume and over a plane cross-section

$$\langle \varphi' \rangle_V = \langle \varphi' \rangle_S, \qquad [1]$$

where

$$\langle \varphi' \rangle_V = \frac{1}{\mathrm{d}V} \int_{\mathrm{d}V} \varphi' \,\mathrm{d}'V, \qquad [2]$$

is the mean volume value, and

$$\langle \varphi' \rangle_S = \frac{1}{\mathrm{d}S} \int_{\mathrm{d}S} \varphi' \,\mathrm{d}'S, \qquad [3]$$

is the mean cross-section value of the variable φ with a local value φ'. Here $\mathrm{d}V$ is the volume of averaging, $\mathrm{d}S$ is the area of averaging; $\mathrm{d}'V$, $\mathrm{d}'S$ are the differentials of volume and area of averaging.

The possibility of use of averaged, or in other words of macroscopical values $\langle \varphi' \rangle_V$ and $\langle \varphi' \rangle_S$, which are associated with the centers of the elements $\mathrm{d}V$ and $\mathrm{d}S$ correspondingly, is named by Nigmatulin (1978) as hypotheses I. Hypotheses II is the statement that these averaged variables are regular functions of the coordinates of mentioned centers and of time, and that their derivatives can be estimated as

$$\frac{\partial\langle\varphi'\rangle_V}{\partial x} \sim \frac{\partial\langle\varphi'\rangle_S}{\partial x} \sim \frac{\varphi_0}{L}, \qquad [4]$$

where φ_0 is the characteristic value of φ (more precisely, its characteristic change—V.N.) in the problem under consideration and L is the length macroscale.

The proof (Nigmatulin 1978) of the statement [1], if the complications connected with consideration of heterogeneous phases separately are ommitted for simplicity, is the following (p. 66 of the book by Nigmatulin): "Let $\mathrm{d}S(x')$ be a family of planes, parallel to each other with cross-sections of volume $\mathrm{d}V$, where $x - \mathrm{d}x \leq x' \leq x + \mathrm{d}x$. It is easy to see that

$$\langle\varphi'\rangle_V = \frac{1}{\mathrm{d}V}\int_{\mathrm{d}V}\varphi'\,\mathrm{d}'V = \frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}\mathrm{d}'x'\int_{\mathrm{d}S}\varphi'\,\mathrm{d}'S$$

$$= \frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}\langle\varphi'\rangle_S\,\mathrm{d}S(x')\,\mathrm{d}'x'. \qquad [5]$$

According to the hypotheses II . . . the variable $\langle\varphi'\rangle_S$ is a regular function, changing slightly over the distance of the order $\mathrm{d}x$. Using the theorem of the mean for the integral, we have

$$\langle\varphi'\rangle_V = \left[\langle\varphi'\rangle_S + O\left(\frac{\partial\langle\varphi'\rangle_S}{\partial x}\,\mathrm{d}x\right)\right]\frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}\mathrm{d}S(x')\,\mathrm{d}'x'$$

$$= \langle\varphi'\rangle_S + \varphi_0 O\left(\frac{\mathrm{d}x}{L}\right) \approx \langle\varphi'\rangle_S, \qquad [6]$$

that is for the representative volumes and surfaces near some point the volume-averaged and plane-averaged . . . values coincide with each other." Hence the equality [1] is thought (Nigmatulin 1978) to be proven and the following conclusions (pp. 66, 82, 83) are made. The tensor of macrostresses $\langle\sigma'^{kl}\rangle_S$ coincides with the tensor $\langle\sigma'^{kl}\rangle_V$. Therefore $\langle\sigma'^{kl}\rangle_S$ is symmetric if the tensor of microstresses is symmetric $\sigma'^{kl} = \sigma'^{lk}$. "The more general averaging, in which the mean surface and the mean volume values are different (the references to papers, Brenner 1970 and Nikolaevskii 1975, are given—V.N.), . . . is only imaginary."

Hence Nigmatulin suggests his own approach to averaging, one that leads to the phenomenological theory (Afanasieff & Nikolaevskii 1969) of mixture with rotating suspended particles in the absence of external moments.

CRUCIAL ROLE OF BALANCE OF MOMENT OF MOMENTUM

Firstly, let us note that even from the cited arguments (Nigmatulin 1978), the equality of the antisymmetrical part of the macrostress tensor to zero $\epsilon_{ikl}\langle\sigma'^{kl}\rangle_S = 0$ does not follow. (Here ϵ_{ikl} is the alternating tensor.) In fact, the equality [5] means that the difference of volume and plane averaged values has the order of the length scale of the elementary macrovolume $\mathrm{d}V$, i.e. $O(\mathrm{d}x/L)$. Therefore this difference is negligible in comparison with the volume-averaged value $\langle\varphi'\rangle_V$.

Let us see where $\epsilon_{ikl}\langle\sigma'^{kl}\rangle_S$ has to be used. It is well-known that antisymmetrical part of macrostresses appears in the balance equation of moment of momentum (see for example, Sedov 1971)

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i v_j}{\partial x_j} = \epsilon_{ikl}\langle\sigma'^{kl}\rangle_S + \frac{\partial \mu_{ij}}{\partial x_j} + c_i, \qquad [7]$$

in addition to the internal moment of momentum U_i, the mean translational velocity v_j, the body-distributed couples c_i, and the couple-stresses μ_{ij}. But the symmetrical part of tensor $\langle\sigma'^{kl}\rangle_S$, which has the order $\langle\sigma'^{kl}\rangle_V$, does not appear in [7]. So we can not neglect the value of order $O(\mathrm{d}x/L)$, because the value of the order $O(1)$ is absent.

The stress $\epsilon_{ikl}\langle\sigma'^{kl}\rangle_S$, which is included in the balance [7] has the order $O(\mathrm{d}x/L)$, that is proportional to the dimension of the elementary macrovolume. But it does not follow that this stress is negligible because the internal moment of momentum includes the specific moment of inertia a^2 of the volume $\mathrm{d}V$. Here, a is the length scale of microstructure, i.e. of a heterogeneous inclusion, an eddy, etc. and $L \gg \mathrm{d}x \gg a \gg \mathrm{d}x'$.

PROPER ANALYSIS OF THE PROBLEM

Let us consider now if the transitions from equality [4] to the equality [1] and from the equality [1] to the estimation [5] are valid. The theorem of the mean of the integral (Heanchin 1953) shows that the integral of the multiplication of two continuous functions over some interval is equal to the multiplication of the integral of one of them and the value of another function at some intermediate point of the interval. However it is not necessary at all that this point is the center of the considered interval (which is necessary in accordance with hypothesis I). Therefore the reference to the theorem on mean value as to the argument for the substitution of the approximate equality [5] for the strict one [1] or in other words as to the argument of transition from the equality [4] to the equality [1] failed.

Moreover, in the interval $x - \mathrm{d}x \le x' \le x + \mathrm{d}x$ the regular† function $\langle\varphi'\rangle_S$ can be represented as the series

$$\langle\varphi'\rangle_S = \langle\varphi'\rangle_S(x) + \frac{\partial\langle\varphi'\rangle_S}{\partial x}(x'-x) + \frac{1}{2}\frac{\partial^2\langle\varphi'\rangle_S}{\partial x^2}(x'-x)^2 + \dots \qquad [8]$$

The substitution of the expansion [8] into the integral [4] will give the following equality

$$\langle\varphi'\rangle_V = \langle\varphi'\rangle_S + \frac{\partial\langle\varphi\rangle_S}{\partial x}\frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}(x'-x)\,\mathrm{d}S(x')\,\mathrm{d}'x' + \frac{1}{2}\frac{\partial^2\langle\varphi\rangle_S}{\partial x^2}\frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}(x'-x)^2\,\mathrm{d}S(x')\,\mathrm{d}'x', \qquad [9]$$

instead of the equality [5].

Which volume has to be selected as the elementary macrovolume $\mathrm{d}V$? In the book by Nigmatulin (1978) this volume is illustrated (figures on pp. 55 and 86) as some spheroids. The results of averaging, however, are introduced into the macroequations formulated in the cartesian coordinate system. But one has to recall now that the differential field equations of continuum mechanics play the role of the true balance equations for a small volume $\mathrm{d}V$, that is in the cartesian coordinates for $\mathrm{d}V = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \mathrm{d}x^3$. Therefore the volume $\mathrm{d}x^3$ is elementary one in the considered case. The calculation of its specific content, i.e. the volume-averaging, is fulfilled over its volume. The change of its content is calculated as a balance of fluxes through its faces, that is over plane oriented cross-sections of differential volume. Therefore $\mathrm{d}S(x') = \mathrm{d}S = \text{const.}$ and the equality [9] can be simplified

$$\langle\varphi'\rangle_V = \langle\varphi'\rangle_S + \frac{1}{2}\frac{\partial^2\langle\varphi'\rangle_S}{\partial x^2}\frac{\mathrm{d}S}{\mathrm{d}V}\frac{2}{3}\mathrm{d}x^3 + \cdots$$

$$= \langle\varphi'\rangle_S + \frac{1}{3}\frac{\partial^2\langle\varphi\rangle_S}{\partial x^2}\mathrm{d}x^2 + \dots. \qquad [10]$$

†This regularity is valid definitely for continuous microfield which is provided by the viscosity of real fluids.

So, the difference of the functions $\langle\varphi'\rangle_V$ and $\langle\varphi'\rangle_S$, defined in the center of mass in the volume dV, has the order of a square of the length scale (not the first power of it). The limit transition $dx \to 0$ is impossible for the averaged regular functions, because the area of averaging decreases to zero also: $dS = dx^2$. The lower boundary for dx is the scale a, i.e. $dx \gg a$. In the continuum mechanics the limit transition from finite differences to differentials is interpreted properly only as $(dx/L) \to 0$ and it is fulfilled because of the increasing of the external scale L of the problem is considered.

Returning to the moments generated by the macrostresses $\epsilon_{ikl}\langle\sigma'^{kl}\rangle_S$ in the balance equation of moment of momentum [7], one can see that they have now exactly the order dx^2 which corresponds to the order of the specific moment of inertia a^2. Because the averaging is fulfilled over the oriented plane dS_l (it is missed in the book by Nigmatulin 1978), it is more convenient to use the symbol (Nikolaevskii 1975) $\langle\sigma'^{kl}\rangle_l$ instead of $\langle\sigma'^{kl}\rangle_S$. Here l is the index of a normal to the plane dS_l. The averaging over oriented planes means that in general $\langle\sigma'^{kl}\rangle_l \neq \langle\sigma'^{kl}\rangle_k$. It follows that the macrostress tensor can be nonsymmetrical although the microstress tensor is symmetrical ($\sigma'^{kl} = \sigma'^{lk}$).

For example, in the suspension of rigid rotating particles U_i is proportional to $na^2\omega$, where n is a number of particles in the elementary differential volume (Afanasieff & Nikolaevskii 1969) and $an \sim dx$. The appearence of the characteristic length a (which is smaller than dx but not equal to zero) in [7] is unavoidable. However, the value U_i is not neglegible in comparison with other terms because the angular velocity ω of a particle can be sufficiently high (Sedov 1971, pp. 152–153).

CONCLUSION

The question of symmetry of macrostresses has to be solved by the consideration of the balance equation of moment of momentum for each particular case (Sedov 1971). The asymmetry of the macrostress tensor presents the possibility to study, for example, inertial relaxation which is generated by the nonequality of angular velocities of suspended particles and of the fluid itself (Afanasieff & Nikolaevskii 1969). The latter is an addition to relaxation effects due to nonequality of translational velocities and temperatures in heterogeneous media (Nikolaevskii 1966).

REFERENCES

AFANASIEFF, E. F. & NIKOLAEVSKII, V. N. 1969 On the development of asymmetric hydrodynamics of suspension with rotating solid particles. In *Problems of Hydrodynamic and Continuity Mechanics* (The 60th Birthday of L. I. Sedov), pp. 16–26. SIAM, Philadelphia.

BRENNER, H. 1970 Rheology of two-phase systems. *Annual Review of Fluid Mechanics*, Vol. 2, pp. 137–176. Annual Rev., Palo Alto, CA.

HEANCHIN, A. JA. 1953 *Short Course in Mathematical Analysis*. Gostechteorizdat, Moscow (In Russian).

KRAIKO, A. N., NIGMATULIN, R. I., STARKOV, V. K. & STRENIN, L. E. 1972 Mechanics of multiphase media. *Itogi Nauki i Tehniki, Gidromehanika* (Sci. Editor NIKOLAEVSKII V. N.), Vol. 6, pp. 93–174. VINITI, Moscow (in Russian).

NIGMATULIN, R. I. 1978 *Foundations of Mechanics of Heterogeneous Media*, 336 pp. Nauka, Moscow (in Russian).

NIKOLAEVSKII, V. N. 1966 On some relaxation processes connected with heterogeneity of continuous media. In *Proc. 11th Int. Cong. Appl. Mech.*, Munich 1964, pp. 862–867. Springer-Verlag, Berlin.

NIKOLAEVSKII, V. N. 1975 Tensor of stresses and averaging in mechanics of continua. *Appl. Math. Mech. (PMM)* **39**, 351–356.

SEDOV, L. I. 1971 *A Course in Continuum Mechanics*, Vol. 1. Walters-Noordhoff, Groningen.